JOURNAL OF COMBINATORIAL THEORY 2, 383-392 (1967)

On a Graph Theorem by Dirac

OYSTEIN ORE

Yale University, New Haven, Connecticut

ABSTRACT

It is shown that the maximal length c of a circuit in a graph satisfies

$$c \geqq 2\varrho_0 + 1, \varrho_0 = \min_a \varrho(a)$$

except for two special classes of graphs.

G. A. Dirac [1] has proved the following theorem: Let G be a finite connected graph with single edges and no separating vertices. Then either G has a Hamilton circuit or the maximal length c of a circuit satisfies

$$[\tfrac{1}{2}c] \geqq \varrho_0,$$

where ϱ_0 is the smallest degree of any vertex. (As usual, the brackets indicate greatest integer.) The theorem has also been proved and extended somewhat by Erdös and Gallai [2] and by Ore [3, Section, 5.5]. In these publications one also finds various applications of the theorem. (See Newman [4] and Ore [5, 6].)

In the present paper we improve on Dirac's theorem in two directions. We begin with a proof of the original theorem, divided into three cases which are convenient also for the proof of both extensions. In Section 2 we improve on the lower bound for c by taking into account the length c_0 of the shortest circuit in G. The remaining sections are devoted to the determination of all graphs in which the longest circuit has the smallest possible length $c = 2\,\varrho_0$. There are two special classes of such graphs. Aside from these one will always have

$$c \geqq 2\,\varrho_0 + 1.$$

1. We denote by

$$C = (a_1, a_2, \ldots, a_c), \tag{1.1}$$

a circuit of maximal length c in G. When we assume that C is not a Hamilton circuit there exists at least one touching edge to C, for instance,

$$E_1 = (b_0, b_1), \qquad \text{where } b_0 = a_c. \tag{1.2}$$

We examine arcs

$$L = (b_0, b_1)(b_1, b_2), \ldots, (b_{e-1}, b_e) \tag{1.3}$$

having only the initial vertex b_0 on C. We assume that L is *completed*, that is, it has been continued as far as possible. Then all edges at b_e must go to vertices on C or L.

CASE 1: $e = 1$. All edges at b_1 go to C. Since C has maximal length there cannot exist edges

$$(b_1, a_i), \; (b_1, a_{i+1})$$

to two successive vertices a_i and a_{i+1} on C. Therefore

$$[c/2] \geqq \varrho(b_1) \geqq \varrho_0. \tag{1.4}$$

CASE 2: $e \geqq 2$ and there are no edges from b_e to C except possibly to $b_0 = a_c$.

Since all edges from b_e go to vertices on L we have

$$e \geqq \varrho(b_e). \tag{1.5}$$

The end-points on L of the edges from b_e we denote by

$$b(j_1), b(j_2), \ldots, b(j_\sigma), \; \sigma = \varrho(b_e) \tag{1.6}$$

with increasing indices

$$j_1 < j_2 < \cdots < j_\sigma = e - 1$$

where possibly $j_1 = 0$.

Since G has no separating vertices it follows from Menger's theorem that there exist two disjoint arcs

$$A_1 = A_1(b_r, a_r), \qquad A_2 = A_2(b', a_s)$$

connecting two vertices b and b' of the arc

$$L(b(j_1), b_e) \tag{1.7}$$

with vertices a_r and a_s on C. (See proof in Ore [3, Chapter 12]). It should be noted that A_1 may reduce to the single vertex b_0. In general, one cannot assert from Menger's theorem that one of the arcs A_1 or A_2 can be taken as $L(b(j_1), b_0)$; however, since this arc exists one may suppose that $b = b(j_1)$. (See the proof just mentioned.)

One of the two arcs $C(a_r, a_s)$ must have a length at least equal to $[\frac{1}{2}(c+1)]$. Furthermore, there exists an arc

$$A_3(b(j_1), b') = L(b(j_1), b(j_t)) + (b(j_t), b_e) + L(b_e, b') \tag{1.8}$$

passing through vertices of L. Here $b(j_t)$ denotes the vertex in (1.6) preceding b'. The length of A_3 is at least σ according to (1.8) (Figure 1).

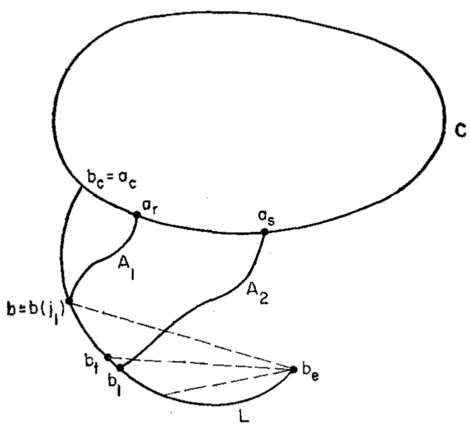

FIGURE 1

By combining the arcs thus defined one obtains a circuit

$$A_1 + C(a_r, a_s) + A_2 + A_3 \tag{1.9}$$

whose length is at least

$$[\tfrac{1}{2}(c+1)] + 1 + \sigma.$$

Since this number cannot exceed c it follows that in Case 2 we have

$$[\tfrac{1}{2} c] \geqq 1 + \sigma = 1 + \varrho(b_e) \geqq 1 + \varrho_0 . \tag{1.10}$$

CASE 3: $e \geqq 2$ and there are edges from b to vertices $a_i \neq a_c$ on C. Such an edge $E = (b_e, a_i)$ creates two circuits

$$\begin{aligned} &C(a_i, a_1, a_c) + L(a_c, b_e) + E \\ &C(a_i, a_{c-1}, a_c) + L(a_c, b_e) + E \end{aligned} \tag{1.11}$$

whose lengths are, respectively,

$$i + e + 1, \quad c - i + e + 1.$$

From the inequalities

$$i + e + 1 \leqq c, \quad c - i + e + 1 \leqq c$$

one obtains

$$e + 1 \leqq i \leqq c - e - 1, \quad c \geqq 2e + 2. \tag{1.12}$$

This shows that there are at most $c - 2e - 1$ available vertices on C for end-points of edges from b_e. Since C is maximal there are no such edges to consecutive vertices on C. Therefore, if $\varrho_C(b_e) = \varrho_C$ is the number of edges from b_e to vertices $a_i \neq a_c$ on C one must have

$$\varrho_C \leqq [\tfrac{1}{2}(c - 2e)] = [\tfrac{1}{2} c] - e. \tag{1.13}$$

On the other hand, the number ϱ_L of edges from b_e to L satisfies

$$\varrho_L \leqq e. \tag{1.14}$$

By addition of these two inequalities follows

$$\varrho_0 \leqq \varrho(b_e) \leqq [\tfrac{1}{2} c].$$

This completes the proof of Dirac's theorem.

2. These estimates can be improved upon by taking into account the length c_0 of the minimal circuits in G.

CASE 1. The vertices on C which are the end-points of edges from b_1 must lie at least $c_0 - 2$ edges apart. This yields

$$\left[\frac{c}{c_0 - 2}\right] \leqq \varrho\,(b_1) \tag{2.1}$$

instead of (1.4).

CASE 2. One finds that the length of the arc A_3 is at least

$$2 + (c_0 - 2)\,(\varrho(b_e) - 2)$$

and this leads to

$$\left[\frac{c}{2}\right] \geqq 3 + (c_0 - 2) \cdot (\varrho(b_e) - 2). \tag{2.2}$$

CASE 3. Instead of (1.13) one obtains

$$\varrho_c \leqq 1 + \left[\frac{c - 2 - 2e}{c_0 - 2}\right]$$

while (1.14) is replaced by

$$e \geqq 1 + (\varrho_L - 1)\,(c_0 - 2).$$

By combining these relations and recalling that $e \geqq 2$ one finds

$$c \geqq (c_0 - 2) \cdot \varrho(b_e) + 9 - 2c_0\,. \tag{2.3}$$

These estimates (2.1), (2.2), and (2.3) are of interest mainly when

$$\varrho_0 \geqq 4, \quad c_0 \geqq 5.$$

In this case (2.3) gives the lowest bound

$$c \geqq (c_0 - 2)\,(\varrho_0 - 2) + 5. \tag{2.4}$$

3. According to Dirac's theorem one has

$$c \geqq 2\,\varrho_0$$

for the lengths of the maximal circuits in a graph without separating vertices. We shall now determine all graphs for which the equality sign holds

$$c = 2\varrho_0\,, \quad \varrho_0 = \min_a \varrho(a) \tag{3.1}$$

We make the preliminary observation that according to the inequality (1.10) Case 2 in Section 1 cannot occur for such graphs.

CASE 1. There exists a vertex b_1 from which all edges go to C.

Since there cannot be such edges to consecutive vertices on C we conclude from (3.1) that one must have

$$\varrho(b_1) = \tfrac{1}{2}\, c = \varrho_0$$

and there is one edge from b_1 to every second vertex on C. Suppose that the edges from b_1 go to the odd-numbered vertices

$$A_1 = \{a_1\,, a_3\,, \ldots, a_{c-1}\}. \tag{3.2}$$

We examine first the edges at the set of even numbered vertices

$$A_2 = \{a_2\,, \ldots, a_c\}. \tag{3.3}$$

No two such vertices can be connected by an edge. Suppose for instance that there exists an edge $(a_c\,, a_{2k})$. One can then construct the circuit

$$(a_c\,, a_{2k}) + C(a_{2k}\,, a_1) + (a_1\,, b_1) + (b_1\,, a_{2k+1}) + C(a_{2k+1}\,, a_c),$$

whose length is $c + 1$. Nor can there be an edge from a vertex in A_2 to a vertex outside of C. Suppose there were such an edge $E_2 = (a_c\,, b_2)$. Not all edges at b_2 can go to C because it would result in a circuit

$$(a_c\,, b_2) + (b_2\,, a_2) + (a_2\,, a_1) + (a_1\,, b_1) + (b_1\,, a_3) + C(a_3\,, a_c)$$

of length $c + 2$. Thus there would exist a returning arc $P(a_c\,, a_k)$ to C beginning in E_2 and having at least the length 3. When k is even we construct the circuit

$$P(a_c\,, a_k) + C(a_k\,, a_1) + (a_1\,, b_1) + (b_1\,, a_{k+1}) + C(a_{k+1}\,, a_c)$$

and for odd k the circuit

$$P(a_c\,, a_k) + C(a_k\,, a_1) + (a_1\,, b_1) + (b_1\,, a_{k+2}) + C(a_{k+1}\,, a_c).$$

Since both of these circuits have lengths greater than c we conclude that no edge E_2 can exist. Thus the edges from the set A_2 all go to vertices in A_1 and

$$\varrho(a_{2i}) = \varrho_0\,.$$

We turn to the edges from vertices in A_1. There can be no such edge, say $E_3 = (a_1, b_2)$, to a vertex b_2 not on C except when all edges from b_2 go to C. If this were not the case there would be a returning arc $P(a_1, a_k)$ to C beginning in E_3 and so also a circuit

$$P(a_1, a_k) + C(a_k, a_3) + (a_3, b_1) + (b_1, a_{k+2}) + C(a_{k+2}, a_1)$$

of length greater than c.

We have established that in our graph no two vertices outside of A_1 are connected by edges and from each of these vertices there are ϱ_0 edges to A_1. We define: For a graph H the *star graph* with r points consists of H and a set R of r vertices outside of H from which there is drawn an edge to each vertex in H. We can then state:

THEOREM 3.1. *In Case* 1 *the graphs satisfying* (3.1) *are the star graphs* $S_r(H)$ *where* H *has the vertex set* A_1 *with* $\frac{1}{2}c$ *vertices and* $r > \frac{1}{2}c$.

We shall define some special graphs before tackling the remaining Case 2. Let

$$U_1(\varrho), \ldots, U_k(\varrho) \tag{3.4}$$

be a family of k complete graphs, each defined on a vertex set A_i with ϱ elements. When any pair of sets A_i have only the fixed vertex a_0 in common we shall call the sum graph

$$\Sigma = \Sigma_i U_i \tag{3.5}$$

the *one-vertex composition* of the graphs (3.4). Suppose next that each pair of the sets A_i have exactly two vertices a_0 and a_1 in common. We then call (3.5) the *two-vertex composition* of the graphs (3.4). In this construction we may or may not include the edge (a_0, a_1) in the sum.

CASE 2. There are no vertices b_1 outside of C whose edges all go to C. Then each edge (a_c, b_1) touching C is the initial edge of a returning path

$$L(a_c, b_1, \ldots, b_e, a_i)$$

to C of length at least 3. When L is taken as a completed arc all edges at b_e must go to L or C. When (3.1) holds, both (1.14) and (1.13) must be equalities. This shows

$$\varrho_C = \varrho_0 - e, \qquad \varrho_L = e,$$

so that

$$\varrho_C + \varrho_L = \varrho(b_e) = \varrho_0 ,$$

and there is an edge from b_e to each vertex on L including a_c. As in (1.12) one must have

$$\varrho_L + 1 \leqq i \leqq 2\,\varrho_0 - \varrho_L - 1$$

for the indices $i \neq c$ for the edges (b_e, a_i) from b_e to C. Since there are just $2\varrho_C - 1$ choices for these vertices a_i they must lie two edges apart on C.

Suppose next that

$$L\,(a_c, b_1, b_e, a_i)$$

is a returning arc to C of maximal length $e + 1$. Then the arcs

$$L\,(a_c, b_e), \qquad L\,(a_i, b_1)$$

can only have edges to C and L at b_e and b_1. Let us show that there can only be a single edge (b_e, a_i) to C. Suppose there were another such edge (b_e, a_{i+2}). Since there is always an edge (b_e, a_c) we see dually that there must be edges

$$(b_1, a_i), \qquad (b_1, a_{i+2}).$$

But then the circuit

$$C\,(a_i, a_c, a_{i+2}) + (a_{i+2}, b_1) + L\,(b_1, b_e) + (b_e, a_i)$$

would have greater length than C.

This observation implies that

$$\varrho\,(b_e) = 1, \qquad \varrho_L\,(b_e) = {}^{a}{}_0 - 1$$

and analogously at b_1. The length of $L\,(a_c, a_i)$ is then ϱ_0 with $i = \varrho_0$.

Consider an arbitrary vertex b_j on L different from the end-points. Through the vertices of $L(a_c, b_e)$ there is an arc from a_c to b_j of length $\varrho_0 - 1$

$$L = L(a_c, b_{j-1}) + (b_{j-1}, b_e) + L(b_e, b_j).$$

As a consequence there is a single edge from b_j to a vertex a_i on C, where $i = \varrho_0$, since otherwise one would have a circuit of greater length than c. The other edges at b_j can only go to vertices on $L(a_c, b_e)$ and so

$\varrho(b_j) = \varrho_0$. Among the edges from a_ϱ and a_c there are $\varrho_0 - 1$ edges to vertices on $L(a_c, a_\varrho)$, but possibly there may be no edge (a_c, a_ϱ). From these observations we conclude that the section graph

$$G(a_c, b_1, ..., b_e, a_\varrho)$$

is either a complete graph $U(\varrho_0)$ on $\varrho_0 + 1$ vertices or such a graph from which the edge (a_c, a_ϱ) has been eliminated. It is attached to the rest of G only at the vertices a_c and a_ϱ.

The maximal returning arc $L(a_c, a_\varrho)$ forms a circuit of length c when combined with any one of the two halves $C(a_\varrho, a_c)$ of C. Then the other half is a maximal returning arc so that it satisfies the conditions just stated. We conclude that the section graph

$$G(a_1, ..., a_c) = U_1(\varrho_0 + 1) + U_2(\varrho_0 + 1) \tag{3.6}$$

is the two-vertex composition graph of two complete graphs on $\varrho_0 + 1$ vertices.

One verifies readily that when there is a returning arc $P(a_i, a_j)$ to the graph (3.6) one must have $i = c$, $j = \varrho_0$, if no circuit of length greater than c is to be created. Suppose that (a_c, b_1) is an edge touching the graph (3.6) at a_c. A completed arc P beginning in this edge can only return to C at a_ϱ; hence there is a single last edge (b_e, a_ϱ). This implies as before that P has the length ϱ_0 so that it is maximal and the preceding conclusions apply to the section graph $G(P)$. This leads to the result:

THEOREM 3.2. *In Case* 3 *the graphs satisfying the condition* (3.1) *are the two-vertex composition of* $k \geqq 3$ *complete graphs on* $\varrho_0 + 1$ *vertices.*

The preceding results can also be formulated as follows:

THEOREM 3.3. *Let G be a finite graph with single edges and no separating vertices. When G is not a graph of the two special types described in Theorems* 3.1 *and* 3.2 *then either G has a Hamilton circuit or the length c of a maximal circuit satisfies*

$$c \geqq 2\,\varrho_0 + 1, \;\; \varrho_0 = \min_a \varrho(a).$$

4. For arcs of maximal length there exists a parallel result to Dirac's theorem (see Ore [3], Theorem 3.4.3):

A finite connected graph with single edges and no loops either has a Hamilton arc or the length l of an arc of maximal length satisfies

$$l \geqq \varrho_0 + \varrho_1 \tag{4.1}$$

where $\varrho_0 \leqq \varrho_1$ are the two smallest local degrees.

An investigation analogous to the preceding may be carried through also for this theorem. The details shall not be given here, but the final result runs as follows:

THEOREM 4.1. *When G has no Hamilton circuit, then*

$$l \geqq \varrho_0 + \varrho_1 + 1$$

except for the two types of graphs:

1. *Star graphs $S_r(H)$ where H has $\varrho + 1$ vertices and $r \geqq \varrho_0 + 2$.*
2. *One-vertex composition of $k \geqq 3$ complete graphs on $\varrho_0 + 1$ vertices.*

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